If the force $F(t)$ is zero for $t < t_0$, then the solution (2.210) will give $x(t) = 0$ for $t < t_0$. This solution is therefore already adjusted to fit the initial condition that the oscillator be at rest before the application of the force. For any other initial condition, a transient given by Eq. (2.133), with appropriate values of $A$ and $\theta$, will have to be added. The solution (2.210) is useful in studying the transient behavior of a mechanical system or electrical circuit when subject to forces of various kinds.

## PROBLEMS

1. a) A certain jet engine at its maximum rate of fuel intake develops a constant thrust (force) of 3000 lb-wt. Given that it is operated at maximum thrust during take-off, calculate the power (in horsepower) delivered to the airplane by the engine when the airplane's velocity is 20 mph, 100 mph, and 300 mph (1 horsepower = 746 watts).
b) A piston engine at its maximum rate of fuel intake develops a constant power of 500 horsepower. Calculate the force it applies to the airplane during take-off at 20 mph, 100 mph, and 300 mph.

2. A particle of mass $m$ is subject to a constant force $F$. At $t = 0$ it has zero velocity. Use the momentum theorem to find its velocity at any later time $t$. Calculate the energy of the particle at any later time from both Eqs. (2.7) and (2.8) and check that the results agree.

3. A particle of mass $m$ is subject to a force given by Eq. (2.192). (In Eq. (2.192), $\delta t$ is a fixed small time interval.) Find the total impulse delivered by the force during the time $-\infty < t < \infty$. If its initial velocity (at $t \to -\infty$) is $v_0$, what is its final velocity (as $t \to \infty$)? Use the momentum theorem.

4. A high-speed proton of electric charge $e$ moves with constant speed $v_0$ in a straight line past an electron of mass $m$ and charge $-e$, initially at rest. The electron is at a distance $a$ from the path of the proton.
a) Assume that the proton passes so quickly that the electron does not have time to move appreciably from its initial position until the proton is far away. Show that the component of force in a direction perpendicular to the line along which the proton moves is

$$F = \frac{e^2 a}{4\pi\epsilon_0 (a^2 + v_0^2 t^2)^{3/2}}, \text{ (mks units)}$$

where $t = 0$ when the proton passes closest to the electron.
b) Calculate the impulse delivered by this force.
c) Write the component of the force in a direction parallel to the proton velocity and show that the net impulse in that direction is zero.
d) Using these results, calculate the (approximate) final momentum and final kinetic energy of the electron.
e) Show that the condition for the original assumption in part (a) to be valid is $(e^2/4\pi\epsilon_0 a) \ll \frac{1}{2}mv_0^2$.

5. A particle of mass $m$ at rest at $t = 0$ is subject to a force $F(t) = F_0 \sin^2 \omega t$.
a) Sketch the form you expect for $v(t)$ and $x(t)$, for several periods of oscillation of the force.
b) Find $v(t)$ and $x(t)$ and compare with your sketch.



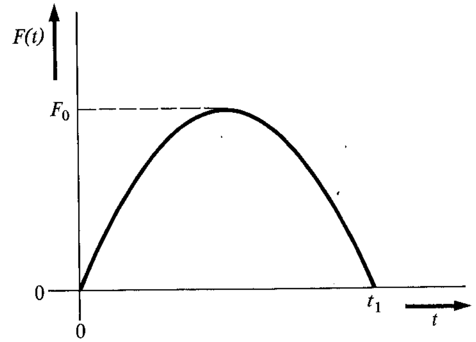


**Fig. 2.9** Force in Problem 6.

6. A particle of mass $m$, initial velocity $v_0$ is subject beginning at $t = 0$ to a force $F(t)$ as sketched in Fig. 2.9.
a) Make a sketch showing $F(t)$ and the expected form of $v(t)$ and $x(t)$.
b) Devise a simple function $F(t)$ having this form, and find $x(t)$ and $v(t)$.

7. A particle which had originally a velocity $v_0$ is subject to a force given by Eq. (2.191).
a) Find $v(t)$ and $x(t)$.
b) Show that as $\delta t \to 0$, the motion approaches motion at constant velocity with an abrupt change in velocity at $t = t_0$ of amount $p_0/m$. ($\delta t$ is a fixed time interval.)

8. A microphone contains a diaphragm of mass $m$ and area $A$, suspended so that it can move freely in a direction perpendicular to the diaphragm. A sound wave impinges on the diaphragm so that the pressure on its front face is

$$p = p_0 + p' \sin \omega t.$$

Assume that the pressure on its back face remains constant at the atmospheric pressure $p_0$. Neglecting all other forces except that due to the pressure difference across the diaphragm, find its motion. In an actual microphone there is a restoring force on the diaphragm which keeps it from moving too far. Since this force is neglected here, nothing prevents the diaphragm from drifting away with a constant velocity. Avoid this difficulty by choosing the initial velocity so that the motion is purely oscillatory. If the output voltage of the microphone is to be proportional to the sound pressure $p'$ and independent of $\omega$, how must it depend upon the amplitude and frequency of the motion of the diaphragm?

9. A tug of war is held between two teams of five men each. Each man weighs 160 lb and can initially pull on the rope with a force of 200 lb-wt. At first the teams are evenly matched, but as the men tire, the force with which each man pulls decreases according to the formula

$$F = (200 \text{ lb-wt})\, e^{-t/\tau},$$

where the mean tiring time $\tau$ is 10 sec for one team and 20 sec for the other. Find the motion. Assume the men do not change their grip on the rope. ($g = 32$ ft-sec$^{-2}$.) What is the final velocity of the two teams? Which of our assumptions is responsible for this unreasonable result?

**10.** A particle initially at rest is subject, beginning at $t = 0$, to a force

$$F = F_0 e^{-\gamma t} \cos(\omega t + \theta).$$

a) Find its motion.
b) How does the final velocity depend on $\theta$, and on $\omega$? [*Hint:* The algebra is simplified by writing $\cos(\omega t+\theta)$ in terms of complex exponential functions.]

**11.** A boat with initial velocity $v_0$ is slowed by a frictional force

$$F = -be^{\alpha v}.$$

a) Find its motion.
b) Find the time and the distance required to stop.

**12.** A boat is slowed by a frictional force $F(v)$. Its velocity decreases according to the formula

$$v = C(t-t_1)^2,$$

where $C$ is a constant and $t_1$ is the time at which it stops. Find the force $F(v)$.

**13.** A jet engine which develops a constant maximum thrust $F_0$ is used to power a plane with a frictional drag proportional to the square of the velocity. If the plane starts at $t = 0$ with a negligible velocity and accelerates with maximum thrust, find its velocity $v(t)$.

**14.** Assume that the engines of a propeller-driven airplane of mass $m$ deliver a constant power $P$ at full throttle. Find the force $F(v)$. Neglecting friction use the method of Section 2.4 to find the velocity and position of the plane as it accelerates down the runway, starting from rest at $t = 0$. Check your result for the velocity using the energy theorem. In what ways are the assumptions in this problem physically unrealistic? In what ways would the answer be changed by more realistic assumptions?

**15.** The engine of a racing car of mass $m$ delivers a constant power $P$ at full throttle. Assuming that the friction is proportional to the velocity, find an expression for $v(t)$ if the car accelerates from a standing start at full throttle. Does your solution behave correctly as $t \to \infty$?

**16.** a) A body of mass $m$ slides on a rough horizontal surface. The coefficient of static friction is $\mu_s$, and the coefficient of sliding friction is $\mu$. Devise an analytic function $F(v)$ to represent the frictional force which has the proper constant value at appreciable velocities and reduces to the static value at very low velocities.
b) Find the motion under the force you have devised if the body starts with an initial velocity $v_0$.

**17.** Find $v(t)$ and $x(t)$ for a particle of mass $m$ which starts at $x_0 = 0$ with velocity $v_0$, subject to a force given by Eq. (2.31) with $n \neq 1$. Find the time to stop, and the distance required to stop, and verify the remarks in the last paragraph of Section 2.4.



**18.** A particle of mass $m$ is subject to a force

$$F = -kx + kx^3/a^2$$

where $k$, $a$ are constants.
a) Find $V(x)$ and discuss the kinds of motion which can occur.
b) Show that if $E = \frac{1}{4}ka^2$ the integral in Eq. (2.46) can be evaluated by elementary methods. Find $x(t)$ for this case, choosing $x_0$, $t_0$ in any convenient way. Show that your result agrees with the qualitative discussion in part (a) for this particular energy.

**19.** A particle of mass $m$ is repelled from the origin by a force inversely proportional to the cube of its distance from the origin. Set up and solve the equation of motion if the particle is initially at rest at a distance $x_0$ from the origin.

**20.** A mass $m$ is connected to the origin with a spring of constant $k$, whose length when relaxed is $l$. The restoring force is very nearly proportional to the amount the spring has been stretched or compressed so long as it is not stretched or compressed very far. However, when the spring is compressed too far, the force increases very rapidly, so that it is impossible to compress the spring to less than half its relaxed length. When the spring is stretched more than about twice its relaxed length, it begins to weaken, and the restoring force becomes zero when it is stretched to very great lengths.
a) Devise a force function $F(x)$ which represents this behavior. (Of course a real spring is deformed if stretched too far, so that $F$ becomes a function of its previous history, but you are to assume here that $F$ depends only on $x$.)
b) Find $V(x)$ and describe the types of motion which may occur.

**21.** A particle of mass $m$ is acted on by a force whose potential energy is

$$V = ax^2 - bx^3.$$

a) Find the force.
b) The particle starts at the origin $x = 0$ with velocity $v_0$. Show that, if $|v_0| < v_c$, where $v_c$ is a certain critical velocity, the particle will remain confined to a region near the origin. Find $v_c$.

**22.** An alpha particle in a nucleus is held by a potential having the shape shown in Fig. 2.10.
a) Describe the kinds of motion that are possible.
b) Devise a function $V(x)$ having this general form and having the values $-V_0$ and $V_1$ at $x = 0$ and $x = \pm x_1$, and find the corresponding force.

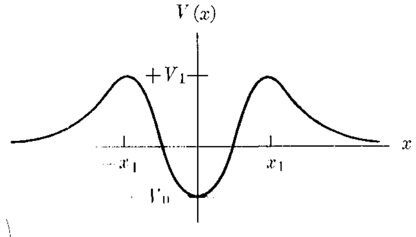


**Fig. 2.10**

**23.** A particle is subject to a force

$$F = -kx + \frac{a}{x^3}.$$

a) Find the potential $V(x)$, describe the nature of the solutions, and find the solution $x(t)$.
b) Can you give a simple interpretation of the motion when $E^2 \gg ka$?

**24.** A particle of mass $m$ is subject to a force given by

$$F = B\left(\frac{a^2}{x^2} - \frac{28a^5}{x^5} + \frac{27a^8}{x^8}\right).$$

The particle moves only along the positive $x$-axis.
a) Find and sketch the potential energy. ($B$ and $a$ are positive.)
b) Describe the types of motion which may occur. Locate all equilibrium points and determine the frequency of small oscillations about any which are stable.
c) A particle starts at $x = 3a/2$ with a velocity $v = -v_0$, where $v_0$ is positive. What is the smallest value of $v_0$ for which the particle may eventually escape to a very large distance? Describe the motion in that case. What is the maximum velocity the particle will have? What velocity will it have when it is very far from its starting point?

**25.** The potential energy for the force between two atoms in a diatomic molecule has the approximate form:

$$V(x) = -\frac{a}{x^6} + \frac{b}{x^{12}},$$

where $x$ is the distance between the atoms and $a$, $b$ are positive constants.
a) Find the force.
b) Assuming one of the atoms is very heavy and remains at rest while the other moves along a straight line, describe the possible motions.
c) Find the equilibrium distance and the period of small oscillations about the equilibrium position if the mass of the lighter atom is $m$.

**26.** Find the solution for the motion of a body subject to a linear repelling force $F = kx$. Show that this is the type of motion to be expected in the neighborhood of a point of unstable equilibrium.

**27.** A particle of mass $m$ moves in a potential well given by

$$V(x) = \frac{-V_0 a^2(a^2 + x^2)}{8a^4 + x^4}.$$

a) Sketch $V(x)$ and $F(x)$.
b) Discuss the motions which may occur. Locate all equilibrium points and determine the frequency of small oscillations about any that are stable.
c) A particle starts at a great distance from the potential well with velocity $v_0$ toward the well. As it passes the point $x = a$, it suffers a collision with another particle, during which it loses a fraction $\alpha$ of its kinetic energy. How large must $\alpha$ be in order that the particle thereafter remains trapped in the well? How large must $\alpha$ be in order that the particle be trapped in one side of the well? Find the turning points of the new motion if $\alpha = 1$.



**28.** Solve Eq. (2.65) by each of the three methods discussed in Sections 2.3, 2.4, and 2.5.

**29.** Derive the solutions (2.74) and (2.75) for a falling body subject to a frictional force proportional to the square of the velocity.

**30.** A body of mass $m$ falls from rest through a medium which exerts a frictional drag (force) $be^{\alpha|v|}$.
a) Find its velocity $v(t)$.
b) What is the terminal velocity?
c) Expand your solution in a power series in $t$, keeping terms up to $t^2$.
d) Why does the solution fail to agree with Eq. (1.28) even for short times $t$?

**31.** A projectile is fired vertically upward with an initial velocity $v_0$. Find its motion, assuming a frictional drag proportional to the square of the velocity. (Constant $g$.)

**32.** Derive equations analogous to Eqs. (2.85) and (2.86) for the motion of a body whose velocity is greater than the escape velocity. [*Hint:* Set $\sinh\beta = (Ex/mMG)^{1/2}$.]

**33.** Find the motion of a body projected upward from the earth with a velocity equal to the escape velocity. Neglect air resistance.

**34.** Starting with $e^{2i\theta} = (e^{i\theta})^2$, obtain formulas for $\sin 2\theta$, $\cos 2\theta$ in terms of $\sin\theta$, $\cos\theta$.

**35.** By writing $\cos\theta$ in the form (2.122) derive the formula

$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta.$$

**36.** Find the general solutions of the equations:

$$\text{a)}\quad m\ddot{x} + b\dot{x} - kx = 0,$$

$$\text{b)}\quad m\ddot{x} - b\dot{x} + kx = 0.$$

Discuss the physical interpretation of these equations and their solutions, assuming that they are the equations of motion of a particle.

**37.** Show that when $\omega_0^2 - \gamma^2$ is very small, the underdamped solution (2.133) is approximately equal to the critically damped solution (2.146), for a short time interval. What is the relation between the constants $C_1$, $C_2$ and $A$, $\theta$? This result suggests how one might discover the additional solution (2.143) in the critical case.

**38.** A freely rolling freight car weighing $10^4$ kg arrives at the end of its track with a speed of 2 m/sec. At the end of the track is a snubber consisting of a firmly anchored spring with $k = 1.6 \times 10^4$ kg/sec$^2$. The car compresses the spring. If the friction is proportional to the velocity, find the damping constant $b_c$ for critical damping. Sketch the motion $x(t)$ and find the maximum distance by which the spring is compressed (for $b = b_c$). Show that if $b \geq b_c$, the car will come to a stop, but if $b \leq b_c$, the car will rebound and roll back down the track. (Note that the car is not fastened to the spring. As long as it pushes on the spring, it moves according to the harmonic oscillator equation, but instead of pulling on the spring, it will simply roll back down the track.)

39. A mass $m$ subject to a linear restoring force $-kx$ and damping $-b\dot{x}$ is displaced a distance $x_0$ from equilibrium and released with zero initial velocity. Find the motion in the underdamped, critically damped, and overdamped cases.

40. Solve Problem 39 for the case when the mass starts from its equilibrium position with an initial velocity $v_0$. Sketch the motion for the three cases.

41. Solve Problem 39 for the case when the mass has an initial displacement $x_0$ and an initial velocity $v_0$ directed back toward the equilibrium point. Show that if $|v_0| > |\gamma_1 x_0|$, the mass will overshoot the equilibrium in the critically damped and overdamped cases so that the remarks at the end of Section 2.9 do not apply. Sketch the motion in these cases.

42. It is desired to design a bathroom scale with a platform deflection of one inch under a 200-lb man. If the motion is to be critically damped, find the required spring constant $k$ and the damping constant $b$. Show that the motion will then be overdamped for a lighter person. If a 200-lb man steps on the scale, what is the maximum upward force exerted by the scale platform against his feet while the platform is coming to rest?

43. A mass of 1000 kg drops from a height of 10 m on a platform of negligible mass. It is desired to design a spring and dashpot on which to mount the platform so that the platform will settle to a new equilibrium position 0.2 m below its original position as quickly as possible after the impact *without overshooting*.
a) Find the spring constant $k$ and the damping constant $b$ of the dashpot. Be sure to examine your proposed solution $x(t)$ to make sure that it satisfies the correct initial conditions and does not overshoot.
b) Find, to two significant figures, the time required for the platform to settle within 1 mm of its final position.

44. A force $F_0 e^{-at}$ acts on a harmonic oscillator of mass $m$, spring constant $k$, and damping constant $b$. Find a particular solution of the equation of motion by starting from the guess that there should be a solution with the same time dependence as the applied force.

45. a) Find the motion of a damped harmonic oscillator subject to a constant applied force $F_0$, by guessing a "steady-state" solution of the inhomogeneous equation (2.91) and adding a solution of the homogeneous equation.
b) Solve the same problem by making the substitution $x' = x - a$, and choosing the constant $a$ so as to reduce the equation in $x'$ to the homogeneous equation (2.90). Hence show that the effect of the application of a constant force is merely to shift the equilibrium position without affecting the nature of the oscillations.

46. An underdamped harmonic oscillator is subject to an applied force

$$F = F_0 e^{-at} \cos(\omega t + \theta).$$



Find a particular solution by expressing $F$ as the real part of a complex exponential function and looking for a solution for $x$ having the same exponential time dependence.

47. An undamped harmonic oscillator ($b = 0$), initially at rest, is subject beginning at $t = 0$ to an applied force $F_0 \sin \omega t$. Find the motion $x(t)$.

48. An undamped harmonic oscillator ($b = 0$) is subject to an applied force $F_0 \cos \omega t$. Show that if $\omega = \omega_0$, there is no steady-state solution. Find a particular solution by starting with a solution for $\omega = \omega_0 + \varepsilon$, and passing to the limit $\varepsilon \to 0$. [*Hint:* If you start with the steady-state solution and let $\varepsilon \to 0$, it will blow up. Try starting with a solution which fits the initial condition $x_0 = 0$, so that it cannot blow up at $t = 0$.]

49. A critically damped harmonic oscillator with mass $m$ and spring constant $k$, is subject to an applied force $F_0 \cos \omega t$. If, at $t = 0$, $x = x_0$ and $v = v_0$, what is $x(t)$?

50. A force $F_0 \cos(\omega t + \theta_0)$ acts on a damped harmonic oscillator beginning at $t = 0$.
a) What must be the initial values of $x$ and $v$ in order that there be no transient?
b) If instead $x_0 = v_0 = 0$, find the amplitude $A$ and phase $\theta$ of the transient in terms of $F_0, \theta_0$.

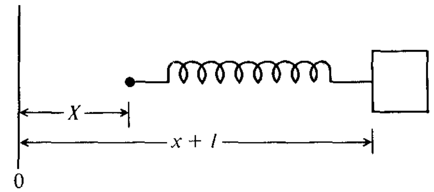


**Fig. 2.11**

51. A mass $m$ is attached to a spring with force constant $k$, relaxed length $l$, as shown in Fig. 2.11. The left end of the spring is not fixed, but is instead made to oscillate with amplitude $a$, frequency $\omega$, so that $X = a \sin \omega t$, where $X$ is measured from a fixed reference point 0. Write the equation of motion, and show that it is equivalent to Eq. (2.148) with an applied force $ka \sin \omega t$, if the friction is given by Eq. (2.31). Show that, if the friction comes instead from a dashpot connected between the ends of the spring, so that the frictional force is $-b(\dot{x} - \dot{X})$, then the equation of motion has an additional applied force $\omega ba \cos \omega t$.

52. An automobile weighing one ton (2000 lb, including passengers but excluding wheels and everything else below the springs) settles one inch closer to the road for every 200 lb of passengers. It is driven at 20 mph over a washboard road with sinusoidal undulations having a distance between bumps of 1 ft and an amplitude of 2 in (height of bumps and depth of holes from mean road level). Find the amplitude of oscillation of the automobile, assuming it moves vertically as a simple harmonic oscillator without damping (no shock absorbers). (Neglect the mass of wheels and springs.) If shock absorbers are added to provide damping, is the ride better or worse? (Use the result of Problem 51.)

53. An undamped harmonic oscillator of mass $m$, natural frequency $\omega_0$, is initially at rest and is subject at $t = 0$ to a blow so that it starts from $x_0 = 0$ with initial velocity $v_0$ and oscillates freely until $t = 3\pi/2\omega_0$. From this time on, a force $F = B \cos(\omega t + \theta)$ is applied. Find the motion.

54. Find the motion of a mass $m$ subject to a restoring force $-kx$, and to a damping force $(\pm)\mu mg$ due to dry sliding friction. Show that the oscillations are isochronous (period independent of amplitude) with the amplitude of oscillation decreasing by $2\mu g/\omega_0^2$ during each half-cycle until the mass comes to a stop. [*Hint:* Use the result of Problem 45. When the force has a different algebraic form at different times during the motion, as here, where the sign of the damping force must be chosen so that the force is always opposed to the velocity, it is necessary to solve the equation of motion separately for each interval of time during which a particular expression for the force is to be used, and to choose as initial conditions for each time interval the final position and velocity of the preceding time interval.]

55. An undamped harmonic oscillator ($\gamma = 0$), initially at rest, is subject to a force given by Eq. (2.191).
a) Find $x(t)$.
b) For a fixed $p_0$, for what value of $\delta t$ is the final amplitude of oscillation greatest?
c) Show that as $\delta t \to 0$, your solution approaches that given by Eq. (2.190).

56. Find the solution analogous to Eq. (2.190) for a critically damped harmonic oscillator subject to an impulse $p_0$ delivered at $t = t_0$.

57. a) Find, using the principle of superposition, the motion of an underdamped oscillator $[\gamma = (1/3)\omega_0]$ initially at rest and subject, after $t = 0$, to a force

$$F = A \sin \omega_0 t + B \sin 3\omega_0 t,$$

where $\omega_0$ is the natural frequency of the oscillator.
b) What ratio of $B$ to $A$ is required in order for the forced oscillation at frequency $3\omega_0$ to have the same amplitude as that at frequency $\omega_0$?

58. A force $F_0(1 - e^{-at})$ acts on a harmonic oscillator which is at rest at $t = 0$. The mass is $m$, the spring constant $k = 4ma^2$, and $b = ma$. Find the motion. Sketch $x(t)$.

*59. Solve Problem 58 for the case $k = ma^2$, $b = 2ma$.

60. Find, by the Fourier-series method, the steady-state solution for the damped harmonic oscillator subject to a force

$$F(t) = \begin{cases} 0, & \text{if } nT < t \le (n+\tfrac{1}{2})T, \\ F_0, & \text{if } (n+\tfrac{1}{2})T < t \le (n+1)T, \end{cases}$$

where $n$ is any integer, and $T = 6\pi/\omega_0$, where $\omega_0$ is the resonance frequency of the oscillator. Show that if $\gamma \ll \omega_0$, the motion is nearly sinusoidal with period $T/3$.

*An asterisk is used, as explained in the Preface, to indicate problems which may be particularly difficult.



61. Find, by the Fourier-series method, the steady-state solution for an undamped harmonic oscillator subject to a force having the form of a rectified sine-wave:

$$F(t) = F_0 \, |\sin \omega_0 t|,$$

where $\omega_0$ is the natural frequency of the oscillator.

62. Solve Problem 58 by using Green's solution (2.210).

63. An underdamped oscillator initially at rest is acted upon, beginning at $t = 0$, by a force given by Eq. (2.191). Find its motion by using Green's solution (2.210).

64. Using the result of Problem 56, find by Green's method the motion of a critically damped oscillator initially at rest and subject to a force $F(t)$.

**5.** Prove the following inequalities. Give a geometric and an algebraic proof (in terms of components) for each:

a) $$|\boldsymbol{A}+\boldsymbol{B}| \le |\boldsymbol{A}|+|\boldsymbol{B}|.$$
b) $$|\boldsymbol{A}\cdot\boldsymbol{B}| \le |\boldsymbol{A}|\,|\boldsymbol{B}|.$$
c) $$|\boldsymbol{A}\times\boldsymbol{B}| \le |\boldsymbol{A}|\,|\boldsymbol{B}|.$$

**6.** a) Obtain a formula analogous to Eq. (3.40) for the magnitude of the sum of three forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3$, in terms of $F_1, F_2, F_3$, and the angles $\theta_{12}, \theta_{23}, \theta_{31}$ between pairs of forces. [Use the suggestions following Eq. (3.40).]
b) Obtain a formula in the same terms for the angle $\alpha_1$, between the total force and the component force $\boldsymbol{F}_1$.

**7.** Prove Eqs. (3.54) and (3.55) from the definition (3.52) of vector differentiation.

**8.** Prove Eqs. (3.56) and (3.57) from the algebraic definition (3.53) of vector differentiation.

**9.** Give suitable definitions, analogous to Eqs. (3.52) and (3.53), for the integral of a vector function $\boldsymbol{A}(t)$ with respect to a scalar $t$:

$$\int_{t_1}^{t_2} \boldsymbol{A}(t)\,dt.$$

Write a set of equations like Eqs. (3.54)–(3.57) expressing the algebraic properties you would expect such an integral to have. Prove that on the basis of either definition

$$\frac{d}{dt}\int_0^t \boldsymbol{A}(t)\,dt = \boldsymbol{A}(t).$$

**10.** A 45° isosceles right triangle $ABC$ has a hypotenuse $AB$ of length $4a$. A particle is acted on by a force attracting it toward a point $O$ on the hypotenuse a distance $a$ from the point $A$. The force is equal in magnitude to $k/r^2$, where $r$ is the distance of the particle from the point $O$. Calculate the work done by this force when the particle moves from $A$ to $C$ to $B$ along the two legs of the triangle. Make the calculation by both methods, that based on Eq. (3.61) and that based on Eq. (3.63).

**11.** A particle moves around a semicircle of radius $R$, from one end $A$ of a diameter to the other $B$. It is attracted toward its starting point $A$ by a force proportional to its distance from $A$. When the particle is at $B$, the force toward $A$ is $F_0$. Calculate the work done against this force when the particle moves around the semicircle from $A$ to $B$.

**12.** A particle is acted on by a force whose components are

$$F_x = ax^3 + bxy^2 + cz,$$
$$F_y = ay^3 + bx^2y,$$
$$F_z = cx.$$

Calculate the work done by this force when the particle moves along a straight line from the origin to the point $(x_0, y_0, z_0)$.



**13.** a) A particle in the $xy$-plane is attracted toward the origin by a force $F = k/y$, inversely proportional to its distance from the $x$-axis. Calculate the work done by the force when the particle moves from the point $x = 0$, $y = a$ to the point $x = 2a$, $y = 0$ along a path which follows the sides of a rectangle consisting of a segment parallel to the $x$-axis from $x = 0$, $y = a$ to $x = 2a$, $y = a$, and a vertical segment from the latter point to the $x$-axis.
b) Calculate the work done by the same force when the particle moves along an ellipse of semiaxes $a$, $2a$. [*Hint:* Set $x = 2a \sin\theta$, $y = a\cos\theta$.]

**14.** Find the $r$- and $\theta$-components of $d\boldsymbol{a}/dt$ in plane polar coordinates, where $\boldsymbol{a}$ is the acceleration of a particle.

**15.** Find the components of $d^2\boldsymbol{A}/dt^2$ in cylindrical polar coordinates, where the vector $\boldsymbol{A}$ is a function of $t$ and is located at a moving point.

**16.** Find the components of $d^3\boldsymbol{r}/dt^3$ in spherical coordinates.

***17.** a) Plane parabolic coordinates $f, h$ are defined in terms of cartesian coordinates $x, y$ by the equations

$$x = f - h, \qquad y = 2(fh)^{1/2},$$

where $f$ and $h$ are never negative. Find $f$ and $h$ in terms of $x$ and $y$. Let unit vectors $\hat{\boldsymbol{f}}, \hat{\boldsymbol{h}}$ be defined in the directions of increasing $f$ and $h$ respectively. That is, $\hat{\boldsymbol{f}}$ is a unit vector in the direction in which a point would move if its $f$-coordinate increases slightly while its $h$-coordinate remains constant. Show that $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{h}}$ are perpendicular at every point. [*Hint:* $\hat{\boldsymbol{f}} = (\hat{\boldsymbol{x}}\,dx + \hat{\boldsymbol{y}}\,dy)[(dx)^2 + (dy)^2]^{-1/2}$, when $df > 0$, $dh = 0$. Why?]
b) Show that $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{h}}$ are functions of $f, h$, and find their derivatives with respect to $f$ and $h$. Show that $\boldsymbol{r} = f^{1/2}(f+h)^{1/2}\hat{\boldsymbol{f}} + h^{1/2}(f+h)^{1/2}\hat{\boldsymbol{h}}$. Find the components of velocity and acceleration in parabolic coordinates.

**18.** A particle moves along the parabola

$$y^2 = 4f_0^2 - 4f_0x,$$

where $f_0$ is a constant. Its speed $v$ is constant. Find its velocity and acceleration components in rectangular and in polar coordinates. Show that the equation of the parabola in polar coordinates is

$$r\cos^2\frac{\theta}{2} = f_0.$$

What is the equation of this parabola in parabolic coordinates (Problem 17)?

**19.** A particle moves with varying speed along an arbitrary curve lying in the $xy$-plane. The position of the particle is to be specified by the distance $s$ the particle has traveled along the curve from some fixed point on the curve. Let $\hat{\boldsymbol{\tau}}(s)$ be a unit vector tangent to the curve at the point $s$ in the direction of increasing $s$. Show that

$$\frac{d\hat{\boldsymbol{\tau}}}{ds} = \frac{\hat{\boldsymbol{\nu}}}{r_0},$$

where $\hat{\nu}(s)$ is a unit vector normal to the curve at the point $s$, and $r(s)$ is the radius of curvature at the point $s$, defined as the distance from the curve to the point of intersection of two nearby normals. Hence derive the following formulas for the velocity and acceleration of the particle:

$$\boldsymbol{v} = \dot{s}\hat{\tau}, \qquad \boldsymbol{a} = \ddot{s}\hat{\tau} + \frac{\dot{s}^2}{r}\hat{\nu}.$$

**20.** Using the properties of the vector symbol $\nabla$, derive the vector identities:

$$\text{curl (curl } A) = \text{grad (div } A) - \nabla^2 A,$$

$$u \text{ grad } v = \text{grad } (uv) - v \text{ grad } u.$$

Then write out the $x$-components of each side of these equations and prove by direct calculation that they are equal in each case. (One must be very careful, in using the first identity in curvilinear coordinates, to take proper account of the dependence of the unit vectors on the coordinates.)

**21.** Calculate curl $A$ in cylindrical coordinates.

**22.** If the particle in Problem 12 moves with a constant velocity $\boldsymbol{v}$, what is the impulse delivered to it by the given force?

**23.** a) Given that the particle in Problem 11 moves with a constant speed $v$ around the semicircle, find the rectangular components $F_x(t)$, $F_y(t)$ of the additional force which must act on it besides the force given in Problem 11. Take the $x$-axis along the diameter AB.

b) Calculate the impulse delivered by this additional force.

**24.** A particle of mass $m$ moves with constant speed $v$ around a circle of radius $r$, starting at $t = 0$ from a point P on the circle. Find the angular momentum about the point P at any time $t$, the force, and the torque about P, and verify that the angular momentum theorem (3.140) is satisfied.

**25.** A particle of mass $m$ moves according to the equations

$$x = x_0 + at^2,$$
$$y = bt^3,$$
$$z = ct.$$

Find the angular momentum $\boldsymbol{L}$ at any time $t$. Find the force $\boldsymbol{F}$ and from it the torque $\boldsymbol{N}$ acting on the particle. Verify that the angular momentum theorem (3.144) is satisfied.

**26.** Give a suitable definition of the angular momentum of a particle about an axis in space. Taking the specified axis as the $z$-axis, express the angular momentum in terms of cylindrical coordinates. If the force acting on the particle has cylindrical components $F_r$, $F_\rho$, $F_\varphi$, prove that the time rate of change of angular momentum about the $z$-axis is equal to the torque about that axis.



**27.** A moving particle of mass $m$ is located by spherical coordinates $r(t)$, $\theta(t)$, $\varphi(t)$. The force acting on it has spherical components $F_r$, $F_\theta$, $F_\varphi$. Calculate the spherical components of the angular momentum vector and of the torque vector about the origin, and verify by direct calculation that the equation

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}$$

follows from Newton's equation of motion.

**28.** The solutions plotted in Fig. 3.28 correspond to the first two of Eqs. (3.151). If $\theta_x = 0$, estimate $\theta_y$ for the case $\omega_x = 2\omega_y$ as drawn. Sketch the corresponding figure for the case $\theta_x = \theta_y$. Sketch a typical figure for the case $4\omega_x = 3\omega_y$.

**29.** Find a lowest order correction to Eq. (3.179) by putting $x_m = (mv_{x0}/b)(1-\delta)$ and solving Eq. (3.175) for $\delta$, assuming $\delta \ll 1$ and $bv_{z0}/mg \gg 1$. [*Hint:* The algebra is not difficult, but you must think carefully about which are the most important terms in this limiting case.]

**30.** Find the maximum height $z_{max}$ reached by a projectile whose equation of motion is Eq. (3.169). Expand your result in a power series in $b$, keeping terms in $z_{max}$ up to first order in $b$, and check the lowest order term against Eq. (3.167).

**31.** A projectile is fired from the origin with initial velocity $\boldsymbol{v}_0 = (v_{x0}, v_{y0}, v_{z0})$. The wind velocity is $\boldsymbol{v}_w = w\hat{\boldsymbol{y}}$. Solve the equations of motion (3.180) for $x$, $y$, $z$ as functions of $t$. Find the point $x_1$, $y_1$ at which the projectile will return to the horizontal plane, keeping only first-order terms in $b$. Show that if air resistance and wind velocity are neglected in aiming the gun, air resistance alone will cause the projectile to fall short of its target a fraction $4bv_{z0}/3mg$ of the target distance, and that the wind causes an additional miss in the $y$-coordinate of amount $2bwv_{z0}^2/(mg^2)$.

**32.** Solve for the next term beyond those given in Eqs. (3.176) and (3.178).

**33.** A projectile is to be fired from the origin in the $xz$-plane ($z$-axis vertical) with muzzle velocity $v_0$ to hit a target at the point $x = x_0$, $z = 0$. (a) Neglecting air resistance, find the correct angle of elevation of the gun. Show that, in general, there are two such angles unless the target is at or beyond the maximum range.

b) Find the first-order correction to the angle of elevation due to air resistance.

**34.** Show that the forces in Problems 11 and 12 are conservative, find the potential energy, and use it to find the work done in each case.

**35.** Determine which of the following forces are conservative, and find the potential energy for those which are:

a) $F_x = 6abz^3y - 20bx^3y^2$, $\quad F_y = 6abxz^3 - 10bx^4y$, $\quad F_z = 18abxz^2y$.
b) $F_x = 18abyz^3 - 20bx^3y^2$, $\quad F_y = 18abxz^3 - 10bx^4y$, $\quad F_z = 6abxyz^2$.
c) $\boldsymbol{F} = \hat{\boldsymbol{x}}F_x(x) + \hat{\boldsymbol{y}}F_y(y) + \hat{\boldsymbol{z}}F_z(z)$.

36. Determine the potential energy for each of the following forces which is conservative:

a) $F_x = 2ax(z^3+y^3)$, $F_y = 2ay(z^3+y^3)+3ay^2(x^2+y^2)$, $F_z = 3az^2(x^2+y^2)$.
b) $F_\rho = a\rho^2 \cos\varphi$, $F_\varphi = a\rho^2 \sin\varphi$, $F_z = 2az^2$.
c) $F_r = -2ar\sin\theta\cos\varphi$, $F_\theta = -ar\cos\theta\cos\varphi$, $F_\varphi = ar\sin\theta\sin\varphi$.

37. Determine the potential energy for each of the following forces which is conservative:

a) $F_x = axe^{-R}$, $F_y = bye^{-R}$, $F_z = cze^{-R}$, where $R = ax^2+by^2+cz^2$.
b) $\boldsymbol{F} = \boldsymbol{A}f(\boldsymbol{A}\cdot\boldsymbol{r})$, where $\boldsymbol{A}$ is a constant vector and $f(s)$ is any suitable function of $s = \boldsymbol{A}\cdot\boldsymbol{r}$.
c) $\boldsymbol{F} = (\boldsymbol{r}\times\boldsymbol{A})f(\boldsymbol{A}\cdot\boldsymbol{r})$.

38. A particle is attracted toward the $z$-axis by a force $\boldsymbol{F}$ proportional to the square of its distance from the $xy$-plane and inversely proportional to its distance from the $z$-axis. Add an additional force perpendicular to $\boldsymbol{F}$ in such a way as to make the total force conservative, and find the potential energy. Be sure to write expressions for the forces and potential energy which are dimensionally consistent.

39. Show that $\boldsymbol{F} = \hat{\boldsymbol{r}}F(r)$ is a conservative force by showing by direct calculation that the integral

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}\cdot d\boldsymbol{r}$$

along any path between $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ depends only on $r_1$ and $r_2$. [*Hint:* Express $\boldsymbol{F}$ and $d\boldsymbol{r}$ in spherical coordinates.]

40. Find the components of force for the following potential-energy functions:

a) $V = axy^2z^3$.
b) $V = \frac{1}{2}kr^2$.
c) $V = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2 + \frac{1}{2}k_z z^2$.

41. Find the force on the electron in the hydrogen molecule ion for which the potential is

$$V = -\frac{e^2}{r_1} - \frac{e^2}{r_2},$$

where $r_1$ is the distance from the electron to the point $y = z = 0$, $x = -a$, and $r_2$ is the distance from the electron to the point $y = z = 0$, $x = a$.

42. Devise a potential-energy function which vanishes as $r \to \infty$, and which yields a force $\boldsymbol{F} = -k\boldsymbol{r}$ when $r \to 0$. Find the force. Verify by doing the appropriate line integrals that the work done by this force on a particle going from $\boldsymbol{r} = 0$ to $\boldsymbol{r} = \boldsymbol{r}_0$ is the same if the particle travels in a straight line as it is if it follows the path shown in Fig. 3.32.

43. The potential energy for an isotropic harmonic oscillator is

$$V = \tfrac{1}{2}kr^2.$$

Plot the effective potential energy for the $r$-motion when a particle of mass $m$ moves with this potential energy and with angular momentum $L$ about the origin. Discuss the types of motion that are possible, giving as complete a description as is possible without carrying out the solution. Find the frequency of revolution for circular motion and the frequency of small radial oscillations about this circular motion. Hence describe the nature of the orbits which differ slightly from circular orbits.



44. Find the frequency of small radial oscillations about steady circular motion for the effective potential given by Eq. (3.232) for an attractive inverse square law force, and show that it is equal to the frequency of revolution.

45. Find $r(t)$, $\theta(t)$ for the orbit of the particle in Problem 43. Compare with the orbits found in Section 3.10 for the three-dimensional harmonic oscillator.

46. A particle of mass $m$ moves under the action of a central force whose potential is

$$V(r) = Kr^4, \qquad K > 0.$$

For what energy and angular momentum will the orbit be a circle of radius $a$ about the origin? What is the period of this circular motion? If the particle is slightly disturbed from this circular motion, what will be the period of small radial oscillations about $r = a$?

47. According to Yukawa's theory of nuclear forces, the attractive force between a neutron and a proton has the potential

$$V(r) = \frac{Ke^{-\alpha r}}{r}, \qquad K < 0.$$

a) Find the force, and compare it with an inverse square law of force.
b) Discuss the types of motion which can occur if a particle of mass $m$ moves under such a force.
c) Discuss how the motions will be expected to differ from the corresponding types of motion for an inverse square law of force.
d) Find $L$ and $E$ for motion in a circle of radius $a$.
e) Find the period of circular motion and the period of small radial oscillations.
f) Show that the nearly circular orbits are almost closed when $a$ is very small.

48. Solve the orbital equation (3.222) for the case $F = 0$. Show that your solution agrees with Newton's first law.

49. It will be shown in Chapter 6 (Problem 7) that the effect of a uniform distribution of dust of density $\rho$ about the sun is to add to the gravitational attraction of the sun on a planet of mass $m$ an additional attractive central force

$$F' = -mkr,$$

where

$$k = \frac{4\pi}{3}\rho G.$$

a) If the mass of the sun is $M$, find the angular velocity of revolution of the planet in a circular orbit of radius $r_0$, and find the angular frequency of small radial oscillations. Hence show that if $F'$ is much less than the attraction due to the sun, a nearly circular orbit will be approximately an ellipse whose major axis precesses slowly with angular velocity

$$\omega_p = 2\pi\rho\left(\frac{r_0^3 G}{M}\right)^{1/2}.$$

b) Does the axis precess in the same or in the opposite direction to the orbital angular velocity? Look up $M$ and the radius of the orbit of Mercury, and calculate the density of dust required to cause a precession of 41 seconds of arc per century.

**50.** a) Discuss by the method of the effective potential the types of motion to be expected for an attractive central force inversely proportional to the cube of the radius:

$$F(r) = -\frac{K}{r^3}, \qquad K > 0.$$

b) Find the ranges of energy and angular momentum for each type of motion.
c) Solve the orbital equation (3.222), and show that the solution is one of the forms:

$$\frac{1}{r} = A\cos\left[\beta(\theta-\theta_0)\right], \tag{1}$$

$$\frac{1}{r} = A\cosh\left[\beta(\theta-\theta_0)\right], \tag{2}$$

$$\frac{1}{r} = A\sinh\left[\beta(\theta-\theta_0)\right], \tag{3}$$

$$\frac{1}{r} = A(\theta-\theta_0), \tag{4}$$

$$\frac{1}{r} = \frac{1}{r_0}e^{\pm\beta\theta}. \tag{5}$$

d) For what values of $L$ and $E$ does each of the above types of motion occur? Express the constants $A$ and $\beta$ in terms of $E$ and $L$ for each case.
e) Sketch a typical orbit of each type.

**51.** (a) Discuss the types of motion that can occur for a central force

$$F(r) = -\frac{K}{r^2}+\frac{K'}{r^3}.$$

Assume that $K > 0$, and consider both signs for $K'$.

b) Solve the orbital equation, and show that the bounded orbits have the form (if $L^2 > -mK'$)

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\alpha\theta},$$

c) Show that this is a precessing ellipse, determine the angular velocity of precession, and state whether the precession is in the same or in the opposite direction to the orbital angular velocity.



**52.** Sputnik I had a perigee (point of closest approach to the earth) 227 km above the earth's surface, at which point its speed was 28,710 km/hr. Find its apogee (maximum) distance from the earth's surface and its period of revolution. (Assume the earth is a sphere, and neglect air resistance. You need only look up $g$ and the earth's radius to do this problem.)

**53.** Explorer I had a perigee 360 km and an apogee 2,549 km above the earth's surface. Find its distance above the earth's surface when it passed over a point 90° around the earth from its perigee.

**54.** A comet is observed a distance of $1.00 \times 10^8$ km from the sun, traveling toward the sun with a velocity of 51.6 km per second at an angle of 45° with the radius from the sun. Work out an equation for the orbit of the comet in polar coordinates with origin at the sun and $x$-axis through the observed position of the comet. (The mass of the sun is $2.00 \times 10^{30}$ kg.)

**55.** It can be shown (Chapter 6, Problems 17 and 21) that the correction to the potential energy of a mass $m$ in the earth's gravitational field, due to the oblate shape of the earth, is approximately, in spherical coordinates, relative to the polar axis of the earth,

$$V' = -\frac{\eta mMGR^2}{5r^3}(1-3\cos^2\theta),$$

where $M$ is the mass of the earth and $2R$, $2R(1-\eta)$ are the equatorial and polar diameters of the earth. Calculate the rate of precession of the perigee (point of closest approach) of an earth satellite moving in a nearly circular orbit in the equatorial plane. Look up the equatorial and polar diameters of the earth, and estimate the rate of precession in degrees per revolution for a satellite 400 miles above the earth.

***56.** Calculate the torque on an earth satellite due to the oblateness potential energy correction given in Problem 55. A satellite moves in a circular orbit of radius $r$ whose plane is inclined so that its normal makes an angle $\alpha$ with the polar axis. Assume that the orbit is very little affected in one revolution, and calculate the average torque during a revolution. Show that the effect of such a torque is to make the normal to the orbit precess in a cone of half angle $\alpha$ about the polar axis, and find a formula for the rate of precession in degrees per revolution. Calculate the rate for a satellite 400 miles above the earth, using suitable values for $M$, $\eta$, and $R$.

**57.** It can be shown that the orbit given by the special theory of relativity for a particle of mass $m$ moving under a potential energy $V(r)$ is the same as the orbit which the particle would follow according to Newtonian mechanics if the potential energy were

$$V(r)-\frac{[E-V(r)]^2}{2mc^2},$$

where $E$ is the energy (kinetic plus potential), and $c$ is the speed of light. Discuss the nature of the orbits for an inverse square law of force according to the theory of relativity. Show by comparing the orbital angular velocity with the frequency of radial oscillations for nearly circular motion that the nearly circular orbits, when the relativistic correction is small, are precessing ellipses, and calculate the angular velocity of precession. [See Eq. (14.101).]

**58.** Mars has a perihelion (closest) distance from the sun of $2.06 \times 10^8$ km, and an aphelion (maximum) distance of $2.485 \times 10^8$ km. Assume that the earth moves in the same plane in a circle of radius $1.49 \times 10^8$ km with a period of one year. From this data alone, find the speed of Mars at perihelion. Assume that a Mariner space probe is launched so that its perihelion is at the earth's orbit and its aphelion at the perihelion of Mars. Find the velocity of the Mariner relative to Mars at the point where they meet. Which has the higher velocity? Which has the higher average angular velocity during the period of the flight?

**59.** Mariner 4 left the earth on an orbit whose perihelion distance from the sun was approximately the distance of the earth ($1.49 \times 10^8$ km), and whose aphelion distance was approximately the distance of Mars from the sun ($2.2 \times 10^8$ km). With what velocity did it leave relative to the earth? With what velocity must it leave the earth (relative to the earth) in order to escape altogether from the sun's gravitational pull? (You need no further data to answer this problem except the length of the year, if you assume the earth moves in a circle.)

**60.** a) A satellite is to be launched from the surface of the earth. Assume the earth is a sphere of radius $R$, and neglect friction with the atmosphere. The satellite is to be launched at an angle $\alpha$ with the vertical, with a velocity $v_0$, so as to coast without power until its velocity is horizontal at an altitude $h_1$ above the earth's surface. A horizontal thrust is then applied by the last stage rocket so as to add an additional velocity $\Delta v_1$ to the velocity of the satellite. The final orbit is to be an ellipse with perigee $h_1$ (point of closest approach) and apogee $h_2$ (point farthest away) measured from the earth's surface. Find the required initial velocity $v_0$ and additional velocity $\Delta v_1$, in terms of $R$, $\alpha$, $h_1$, $h_2$, and $g$, the acceleration of gravity at the earth's surface.

b) Write a formula for the change $\delta h_1$ in perigee height due to a small error $\delta\beta$ in the final thrust direction, to order $(\delta\beta)^2$.

**61.** Two planets move in the same plane in circles of radii $r_1$, $r_2$ about the sun. A space probe is to be launched from planet 1 with velocity $v_1$ relative to the planet, so as to reach the orbit of planet 2. (The velocity $v_1$ is the relative velocity after the probe has escaped from the gravitational field of the planet.) Show that $v_1$ is a minimum for an elliptical orbit whose perihelion and aphelion are $r_1$ and $r_2$. In that case, find $v_1$, and the relative velocity $v_2$ between the space probe and planet 2 if the probe arrives at radius $r_2$ at the proper time to intercept planet 2. Express your results in terms of $r_1$, $r_2$, and the length of the year $Y_1$ of planet 1. Look up the appropriate values of $r_1$ and $r_2$, and estimate $v_1$ for trips to Venus and Mars from the earth.

**62.** A rocket is in an elliptical orbit around the earth, perigee $r_1$, apogee $r_2$, measured from the center of the earth. At a certain point in its orbit, its engine is fired for a short time so as to give a velocity increment $\Delta v$ in order to put the rocket on an orbit which escapes from the earth with a final velocity $v_0$ relative to the earth. (Neglect any effects due to the sun and moon.) Show that $\Delta v$ is a minimum if the thrust is applied at perigee, parallel to the orbital velocity. Find $\Delta v$ in that case in terms of the elliptical orbit parameters $\varepsilon$, $a$, the acceleration $g$ at a distance $R$ from the earth's center, and the final velocity $v_0$. Can you explain physically why $\Delta v$ is smaller for larger $\varepsilon$?

**63.** A satellite moves around the earth in an orbit which passes across the poles. The time



at which it crosses each parallel of latitude is measured so that the function $\theta(t)$ is known. Show how to find the perigee, the semimajor axis, and the eccentricity of its orbit in terms of $\theta(t)$, and the value of $g$ at the surface of the earth. Assume the earth is a sphere of radius $R$.

**64.** A particle of mass $m$ moves in an elliptical orbit of major axis $2a$, eccentricity $\varepsilon$, in such a way that the radius to the particle from the center of the ellipse sweeps out area at a constant rate

$$\frac{dS}{dt} = C,$$

and with period $\tau$ independent of $a$ and $\varepsilon$. (a) Write out the equation of the ellipse in polar coordinates with origin at the center of the ellipse.

b) Show that the force on the particle is a central force, and find $F(r)$ in terms of $m$, $\tau$.

**65.** Show that the Rutherford cross-section formula (3.276) holds also when one of the charges is negative.

**66.** A particle is reflected from the surface of a hard sphere of radius $R$ in such a way that the incident and reflected lines of travel lie in a common plane with the radius to the point of impact and make equal angles with the radius. Find the cross-section $d\sigma$ for scattering through an angle between $\Theta$ and $\Theta + d\Theta$. Integrate $d\sigma$ over all angles and show that the total cross-section has the expected value $\pi R^2$.

**67.** Exploit the analogy $u, \theta \leftrightarrow x, t$ between Eqs. (3.222) and (2.39) in order to develop a solution of Eq. (3.222) analogous to the solution (2.46) of Eq. (2.39). Use your solution to show that the scattering angle $\Theta$ (Fig. 3.42) for a particle subject to a central force $F(r)$ is given by

$$\Theta = \left|\pi - 2s \int_0^{u_0} \left[1 - s^2u^2 - V(u^{-1})/(\tfrac{1}{2}mv_0^2)\right]^{-1/2} du\right|,$$

where $V(r = u^{-1})$ is the potential energy,

$$V(r) = \int_r^{\infty} F(r)\, dr,$$

$s$ is the impact parameter, and $u_0$ is the value of $u$ at which the quantity in square brackets vanishes. [This problem is not difficult if you keep clearly in mind the physical and geometrical significance of the various quantities involved at each step in the solution.]

**68.** Show that a hard sphere as defined in Problem 66 can be represented as a limiting case of a central force where

$$V(r) = \begin{cases} 0, & \text{if } r > R, \\ \infty, & \text{if } r < R, \end{cases}$$

that is, show that such a potential gives the same law of reflection as specified in Problem 66. Hence use the result of Problem 67 to solve Problem 66.

**69.** Use the result of Problem 67 to derive the Rutherford cross-section formula (3.276).

**70.** A rocket moves with initial velocity $v_0$ toward the moon of mass $M$, radius $r_0$. Find the cross section $\sigma$ for striking the moon. Take the moon to be at rest, and ignore all other bodies.

**71.** Show that for a repulsive central force inversely proportional to the cube of the radius,

$$F(r) = \frac{K}{r^3}, \qquad K > 0,$$

the orbits are of the form (1) given in Problem 50, and express $\beta$ in terms of $K$, $E$, $L$, and the mass $m$ of the incident particle. Show that the cross-section for scattering through an angle between $\Theta$ and $\Theta + d\Theta$ for a particle subject to this force is

$$d\sigma = \frac{2\pi^3 K}{mv_0^2}\frac{\pi - \Theta}{\Theta^2(2\pi - \Theta)^2}\, d\Theta.$$

**72.** A particle of charge $q$, mass $m$ at rest in a constant, uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ is subject, beginning at $t = 0$, to an oscillating electric field

$$\boldsymbol{E} = E_0\hat{\boldsymbol{x}} \sin \omega t.$$

Find its motion.

**73.** Solve Problem 72 for the case $\omega = qB_0/mc$.

**74.** A charged particle moves in a constant, uniform electric and magnetic field. Show that if we introduce a new variable

$$\boldsymbol{r}' = \boldsymbol{r} - \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} ct,$$

the equation of motion for $\boldsymbol{r}'$ is the same as that for $\boldsymbol{r}$ except that the component of $\boldsymbol{E}$ perpendicular to $\boldsymbol{B}$ has been eliminated.

**75.** A particle of charge $q$ in a cylindrical magnetron moves in a uniform magnetic field

$$\boldsymbol{B} = B\hat{\boldsymbol{z}},$$

and an electric field, directed radially outward or inward from a central wire along the $z$-axis,

$$\boldsymbol{E} = \frac{a}{\rho}\hat{\boldsymbol{\rho}},$$

where $\rho$ is the distance from the $z$-axis, and $\hat{\boldsymbol{\rho}}$ is a unit vector directed radially outward from the $z$-axis. The constants $a$ and $B$ may be either positive or negative.

a) Set up the equations of motion in cylindrical coordinates.

b) Show that the quantity

$$m\rho^2\dot{\varphi} + \frac{qB}{2c}\rho^2 = K$$

is a constant of the motion.

c) Using this result, give a qualitative discussion, based on the energy integral, of the types of motion that can occur. Consider all cases, including all values of $a$, $B$, $K$, and $E$.

d) Under what conditions can circular motion about the axis occur?

e) What is the frequency of small radial oscillations about this circular motion?



**76.** A velocity selector for a beam of charged particles of mass $m$, charge $e$, is to be designed to select particles of a particular velocity $v_0$. The velocity selector utilizes a uniform electric field $E$ in the $x$-direction and a uniform magnetic field $B$ in the $y$-direction. The beam emerges from a narrow slit along the $y$-axis and travels in the $z$-direction. After passing through the crossed fields for a distance $l$, the beam passes through a second slit parallel to the first and also in the $yz$-plane. The fields $E$ and $B$ are chosen so that particles with the proper velocity moving parallel to the $z$-axis experience no net force.

a) If a particle leaves the origin with a velocity $v_0$ at a small angle with the $z$-axis, find the point at which it arrives at the plane $z = l$. Assume that the initial angle is small enough so that second-order terms in the angle may be neglected.

b) What is the best choice of $E$, $B$ in order that as large a fraction as possible of the particles with velocity $v_0$ arrive at the second slit, while particles of other velocities miss the slit as far as possible?

c) If the slit width is $h$, what is the maximum velocity deviation $\delta v$ from $v_0$ for which a particle moving initially along the $z$-axis can pass through the second slit? Assume that $E$, $B$ have the values chosen in part (b).

slides over the other, as in Fig. 4.15. We assume that the force of friction is proportional to the relative velocity of the two masses. The equations of motion of $m_1$ and $m_2$ are then

$$m_1\ddot{x}_1 = -k_1x_1 - b(\dot{x}_1+\dot{x}_2), \tag{4.184}$$

$$m_2\ddot{x}_2 = -k_2x_2 - b(\dot{x}_2+\dot{x}_1), \tag{4.185}$$

or

$$m_1\ddot{x}_1 + b\dot{x}_1 + k_1x_1 + b\dot{x}_2 = 0, \tag{4.186}$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + k_2x_2 + b\dot{x}_1 = 0. \tag{4.187}$$

The coupling is expressed in Eqs. (4.186), (4.187) by a term in the equation of motion of each oscillator depending on the velocity of the other. The oscillators may also be coupled by a mass, as in Fig. 4.16. It is left to the reader to set up the equations of motion. (See Problem 40 at the end of this chapter.)

Two oscillators may be coupled in such a way that the force acting on one depends on the position, velocity, or acceleration of the other, or on any combination of these. In general, all three types of coupling occur to some extent; a spring, for example, has always some mass, and is subject to some internal friction. Thus the most general pair of equations for two coupled harmonic oscillators is of the form

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + m_c\ddot{x}_2 + b_c\dot{x}_2 + k_cx_2 = 0, \tag{4.188}$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 + m_c\ddot{x}_1 + b_c\dot{x}_1 + k_cx_1 = 0. \tag{4.189}$$

These equations can be solved by the method described above, with similar results. Two normal modes of vibration appear, if the frictional forces are not too great.

Equations of the form (4.188), (4.189), or the simpler special cases considered in the preceding discussions, arise not only in the theory of coupled mechanical oscillators, but also in the theory of coupled electrical circuits. Applying Kirchhoff's second law to the two meshes of the circuit shown in Fig. 4.17, with mesh currents

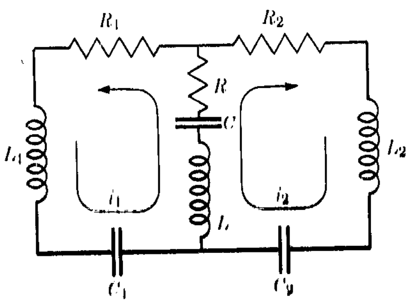


**Fig. 4.17** Coupled oscillating circuits.



$i_1$, $i_2$ around the two meshes as shown, we obtain

$$(L+L_1)\ddot{q}_1 + (R+R_1)\dot{q}_1 + \left(\frac{1}{C}+\frac{1}{C_1}\right)q_1 + L\ddot{q}_2 + R\dot{q}_2 + \frac{1}{C}q_2 = 0, \tag{4.190}$$

and

$$(L+L_2)\ddot{q}_2 + (R+R_2)\dot{q}_2 + \left(\frac{1}{C}+\frac{1}{C_2}\right)q_2 + L\ddot{q}_1 + R\dot{q}_1 + \frac{1}{C}q_1 = 0, \tag{4.191}$$

where $q_1$ and $q_2$ are the charges built up on $C_1$ and $C_2$ by the mesh currents $i_1$ and $i_2$. These equations have the same form as Eqs. (4.188), (4.189), and can be solved by similar methods. In electrical circuits, the damping is often fairly large, and finding the solution becomes a formidable task.

The discussion of this section can be extended to the case of any number of coupled mechanical or electrical harmonic oscillators, with analogous results. The algebraic details become almost prohibitive, however, unless we make use of more advanced mathematical techniques. We therefore postpone further discussion of this problem to Chapter 12.

All mechanical and electrical vibration problems reduce in the limiting case of small amplitudes of vibration to problems involving one or several coupled harmonic oscillators. Problems involving vibrations of strings, membranes, elastic solids, and electrical and acoustical vibrations in transmission lines, pipes, or cavities, can be reduced to problems of coupled oscillators, and exhibit similar normal modes of vibration. The treatment of the behavior of an atom or molecule according to quantum mechanics results in a mathematical problem identical with the problem of coupled harmonic oscillators, in which the energy levels play the role of oscillators, and external perturbing influences play the role of the coupling mechanism.

## PROBLEMS

**1.** Formulate and prove a conservation law for the angular momentum about the origin of a system of particles confined to a plane.

**2.** Water is poured into a barrel at the rate of 120 lb per minute from a height of 16 ft. The barrel weighs 25 lb, and rests on a scale. Find the scale reading after the water has been pouring into the barrel for one minute.

**3.** A ballistic pendulum to be used to measure the speed of a bullet is constructed by suspending a block of wood of mass $M$ by a cord of length $l$. The pendulum initially hangs vertically at rest. A bullet of mass $m$ is fired into the block and becomes imbedded in it. The pendulum then begins to swing and rises until the cord makes a maximum angle $\theta$ with the vertical. Find the initial speed of the bullet in terms of $M$, $m$, $l$, and $\theta$ by applying appropriate conservation laws.

**4.** A box of mass $m$ falls on a conveyor belt moving with constant speed $v_0$. The coefficient of sliding friction between the box and the belt is $\mu$. How far does the box slide along the belt before it is moving with the same speed as the belt? What force $F$ must be applied to the belt to keep it moving at constant speed after the box falls on it, and for how long? Calculate the impulse delivered by this force and check that momentum is conserved between the time before the box falls on the belt and the time when the box is moving with the belt. Calculate the work done by the force $F$ in pulling the belt. Calculate the work dissipated in friction between the box and the belt. Check that the energy delivered to the belt by the force $F$ is just equal to the kinetic energy increase of the box plus the energy dissipated in friction.

**5.** A scoop of mass $m_1$ is attached to an arm of length $l$ and negligible weight. The arm is pivoted so that the scoop is free to swing in a vertical arc of radius $l$. At a distance $l$ directly below the pivot is a pile of sand. The scoop is lifted until the arm is at a 45° angle with the vertical, and released. It swings down and scoops up a mass $m_2$ of sand. To what angle with the vertical does the arm of the scoop rise after picking up the sand? This problem is to be solved by considering carefully which conservation laws are applicable to each part of the swing of the scoop. Friction is to be neglected, except that required to keep the sand in the scoop.

**6.** a) A spherical satellite of mass $m$, radius $a$, moves with speed $\mathbf{v}$ through a tenuous atmosphere of density $\rho$. Find the frictional force on it, assuming that the speed of the air molecules can be neglected in comparison with $\mathbf{v}$, and that each molecule which is struck becomes embedded in the skin of the satellite. Do you think these assumptions are valid?

b) If the orbit is a circle 400 km above the earth (radius 6360 km), where $\rho = 10^{-11}\mathrm{kg/m^3}$, and if $a = 1$ m, $m = 100$ kg, find the change in altitude and the change in period of revolution in one week.

**7.** A lunar landing craft approaches the moon's surface. Assume that one-third of its weight is fuel, that the exhaust velocity from its rocket engine is 1500 m/sec, and that the acceleration of gravity at the lunar surface is one-sixth of that at the earth's surface. How long can the craft hover over the moon's surface before it runs out of fuel?

**8.** A toy rocket consists of a plastic bottle partly filled with water containing also air at a high pressure $p$. The water is ejected through a small nozzle of area $A$. Calculate the exhaust velocity $v$ by assuming that frictional losses of energy are negligible, so that the kinetic energy of the escaping water is equal to the work done by the gas pressure in pushing it out. Show that the thrust of this rocket engine is then $2\,pA$. (Assume that the water leaves the nozzle of area $A$ with velocity $v$.) If the empty rocket weighs 500 g, if it contains initially 500 g of water, and if $A = 5\ \mathrm{mm}^2$, what pressure is required in order that the rocket can just support itself against gravity? If it is then released so that it accelerates upward, what maximum velocity will it reach? Approximately how high will it go? What effects are neglected in the calculation, and how would each of them affect the final result?

***9.** A two-stage rocket is to be built capable of accelerating a 100-kg payload to a velocity of 6000 m/sec in free flight in empty space (no gravitational field). (In a two-stage rocket, the first stage is detached after exhausting its fuel, before the second stage is fired.) Assume that the fuel used can reach an exhaust velocity of 1500 m/sec, and that structural requirements imply that an empty rocket (without fuel or payload) will weigh 10% as much as the fuel it can carry. Find the optimum choice of masses for the two stages so that the total take-off weight is a minimum. Show that it is impossible to build a single-stage rocket which will do the job.



**10.** A rocket is to be fired vertically upward. The initial mass is $M_0$, the exhaust velocity $-u$ is constant, and the rate of exhaust $-(dM/dt) = A$ is constant. After a total mass $\Delta M$ is exhausted, the rocket engine runs out of fuel.

a) Neglecting air resistance and assuming that the acceleration $g$ of gravity is constant, set up and solve the equation of motion.

*b) Show that if $M_0$, $u$, and $\Delta M$ are fixed, then the larger the rate of exhaust $A$, that is, the faster it uses up its fuel, the greater the maximum altitude reached by the rocket.

**11.** Assume that essentially all of the mass $M$ of the gyroscope in Fig. 4.1 is concentrated in the rim of the wheel of radius $R$, and that the center of mass lies on the axis at a distance $l$ from the pivot point $Q$. If the gyroscope rotates rapidly with angular velocity $\omega$, show that the angular velocity of precession of its axis in a cone making an angle $\alpha$ with the vertical is approximately

$$\omega_p = gl/(R^2\omega^2).$$

**12.** A diver executing a $2\frac{1}{2}$ flip doubles up with his knees in his arms in order to increase his angular velocity. Estimate the ratio by which he thus increases his angular velocity relative to his angular velocity when stretched out straight with his arms over his head. Explain your reasoning.

**13.** A uniform spherical planet of radius $a$ revolves about the sun in a circular orbit of radius $r_0$, and rotates about its axis with angular velocity $\omega_0$, normal to the plane of the orbit. Due to tides raised on the planet by the sun, its angular velocity of rotation is decreasing. Find a formula expressing the orbit radius $r$ as a function of angular velocity $\omega$ of rotation at any later or earlier time. [You will need formulas (5.9) and (5.91) from Chapter 5.] Apply your formula to the earth, neglecting the effect of the moon, and estimate how much farther the earth will be from the sun when the day has become equal to the present year. If the effect of the moon were taken into account, would the distance be greater or less?

***14.** A mass $m$ of gas and debris surrounds a star of mass $M$. The radius of the star is negligible in comparison with the distances to the particles of gas and debris. The material surrounding the star has initially a total angular momentum $\mathbf{L}$, and a total kinetic and potential energy $E$. Assume that $m \ll M$, so that the gravitational fields due to the mass $m$ are negligible in comparison with that of the star. Due to internal friction, the surrounding material continually loses mechanical energy. Show that there is a maximum energy $\Delta E$ which can be lost in this way, and that when this energy has been lost, the material must all lie on a circular ring around the star (but not necessarily uniformly distributed). Find $\Delta E$ and the radius of the ring. (You will need to use the method of Lagrange multipliers.)

**15.** A particle of mass $m_1$, energy $T_{1I}$ collides elastically with a particle of mass $m_2$, at rest. If the mass $m_2$ leaves the collision at an angle $\vartheta_2$ with the original direction of motion of $m_1$, what is the energy $T_{2F}$ delivered to particle $m_2$? Show that $T_{2F}$ is a maximum for a head-on collision, and that in this case the energy lost by the incident particle in the collision is

$$T_{1I} - T_{1F} = \frac{4m_1m_2}{(m_1 + m_2)^2}\,T_{1I}.$$

eliminated from Eq. (5.190) by means of Eq. (5.187). If we eliminate the density, we have

$$\frac{dp}{dz} = -\frac{Mg}{RT}p. \tag{5.192}$$

As an example, if we assume that the atmosphere is uniform in temperature and composition, we can solve Eq. (5.192) for the atmospheric pressure as a function of altitude:

$$p = p_0 \exp\left(-\frac{Mg}{RT}z\right). \tag{5.193}$$

## PROBLEMS

1. (a) Prove that the total kinetic energy of the system of particles making up a rigid body, as defined by Eq. (4.37), is correctly given by Eq. (5.16) when the body rotates about a fixed axis.
(b) Prove that the potential energy given by Eq. (5.14) is the total work done against the external forces when the body is rotated from $\theta_s$ to $\theta$, if $N_z$ is the sum of the torques about the axis of rotation due to the external forces.

2. Using the scheme of analogy in Section 5.2, formulate a theorem analogous to that given by Eq. (2.8) and prove it, starting from Eq. (5.13).

3. Prove, starting with the equation of motion (5.13) for rotation, that if $N_z$ is a function of $\theta$ alone, then $T + V$ is constant.

4. The balance wheel of a watch consists of a ring of mass $M$, radius $a$, with spokes of negligible mass. The hairspring exerts a restoring torque $N_z = -k\theta$. Find the motion if the balance wheel is rotated through an angle $\theta_0$ and released.

5. A wheel of mass $M$, radius of gyration $k$, spins smoothly on a *fixed* horizontal axle of radius $a$ which passes through a hole of slightly larger radius at the hub of the wheel. The coefficient of friction between the bearing surfaces is $\mu$. If the wheel is initially spinning with angular velocity $\omega_0$, find the time and the number of turns that it takes to stop.

6. A wheel of mass $M$, radius of gyration $k$ is mounted on a horizontal axle. A coiled spring attached to the axle exerts a torque $N = -K\theta$ tending to restore the wheel to its equilibrium position $\theta = 0$. A mass $m$ is located on the rim of the wheel at distance $2k$ from the axle at a point vertically above the axle when $\theta = 0$. Describe the kinds of motion which can occur, locate the positions of stable or unstable equilibrium of the wheel if any, and find the frequencies of small oscillations about the equilibrium points. Consider two cases: (a) $K > 2mgk$, (b) $K = 4mgk/\pi$. What if $K < 4mgk/5\pi$? [*Hint:* Solve the trigonometric equation graphically.]

7. An airplane propeller of moment of inertia $I$ is subject to a driving torque

$$N = N_0(1 + \alpha \cos \omega_0 t),$$

and to a frictional torque due to air resistance

$$N_f = -b\dot{\theta}.$$

Find its steady state motion.



8. A motor armature weighing 2 kg has a radius of gyration of 5 cm. Its no-load speed is 1500 rpm. It is wound so that its torque is independent of its speed. At full speed, it draws a current of 2 amperes at 110 volts. Assume that the electrical efficiency is 80%, and that the friction is proportional to the square of the angular velocity. Find the time required for it to come up to a speed of 1200 rpm after being switched on without load.

9. Derive Eqs. (5.35) and (5.36).

10. Assume that a simple pendulum suffers a frictional torque $-mb_1\dot{\theta}$ due to friction at the point of support, and a frictional force $-b_2\boldsymbol{v}$ on the bob due to air resistance, where $\boldsymbol{v}$ is the velocity of the bob. The bob has a mass $m$, and is suspended by a string of length $l$. Find the time required for the amplitude to damp to $1/e$ of its initial (small) value. How should $m$, $l$ be chosen if it is desired that the pendulum swing as long a time as possible? How should $m$, $l$ be chosen if it is desired that the pendulum swing through as many cycles as possible?

11. A child of mass $m$ sits in a swing of negligible mass suspended by a rope of length $l$. Assume that the dimensions of the child are negligible compared with $l$. His father pulls the child back until the rope makes an angle of one radian with the vertical, then pushes with a force $F = mg$ along the arc of a circle until the rope is vertical and releases the swing. (a) How high up will the swing go? (b) For what length of time did the father push on the swing? (Assume that it is permissible to write $\sin\theta \doteq \theta$ for $\theta < 1$.) Compare with the time required for the swing to reach the vertical if he simply releases the swing without pushing on it.

12. A baseball bat held horizontally at rest is struck at a point $O'$ by a ball which delivers a horizontal impulse $J'$ perpendicular to the bat. Let the bat be initially parallel to the $x$-axis, and let the baseball be traveling in the negative direction parallel to the $y$-axis. The center of mass $G$ of the bat is initially at the origin, and the point $O'$ is at a distance $h'$ from $G$. Assuming that the bat is let go just as the ball strikes it, and neglecting the effect of gravity, calculate and sketch the motion $x(t)$, $y(t)$ of the center of mass, and also of the center of percussion, during the first few moments after the blow, say until the bat has rotated a quarter turn. Comment on the difference between the initial motion of the center of mass and that of the center of percussion.

13. A compound pendulum is arranged to swing about either of two parallel axes through two points $O$, $O'$ located on a line through the center of mass. The distances $h$, $h'$ from $O$, $O'$ to the center of mass, and the periods $\tau$, $\tau'$ of small amplitude vibrations about the axes through $O$ and $O'$ are measured. $O$ and $O'$ are arranged so that each is approximately the center of oscillation relative to the other. Given $\tau = \tau'$, find a formula for $g$ in terms of measured quantities. Given that $\tau' = \tau(1+\delta)$, where $\delta \ll 1$, find a correction to be added to your previous formula so that it will be correct to terms of order $\delta$.

14. Prove that if a body is composed of two or more parts whose centers of mass are known, then the center of mass of the composite body can be computed by regarding its component parts as single particles located at their respective centers of mass. Assume that each component part $k$ is described by a density $\rho_k(\boldsymbol{r})$ of mass continuously distributed over the region occupied by part $k$.

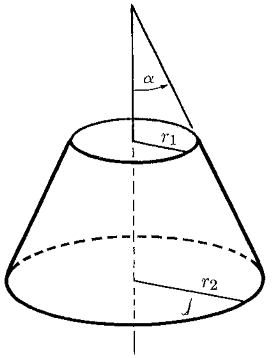


**Fig. 5.28** Frustum of a cone.

**15.** A circular disk of radius $a$ lies in the $xy$-plane with its center at the origin. The half of the disk above the $x$-axis has a density $\sigma$ per unit area, and the half below the $x$-axis has a density $2\sigma$. Find the center of mass $G$, and the moments of inertia about the $x$-, $y$-, and $z$-axes, and about parallel axes through $G$. Make as much use of labor-saving theorems as possible.

**16.** (a) Work out a formula for the moments of inertia of a cone of mass $m$, height $h$, and generating angle $\alpha$, about its axis of symmetry, and about an axis through the apex perpendicular to the axis of symmetry. Find the center of mass of the cone.
b) Use these results to determine the center of mass of the frustum of a cone, shown in Fig. 5.28, and to calculate the moments of inertia about horizontal axes through each base and through the center of mass. The mass of the frustum is $M$.

**17.** Find the moments of inertia of the block shown in Fig. 5.8, about axes through its center of mass parallel to each of the three edges of the block.

**18.** Through a sphere of mass $M$, radius $R$, a plane saw cut is made at a distance $\frac{1}{2}R$ from the center. The smaller piece of the sphere is discarded. Find the center of mass of the remaining piece, and the moments of inertia about its axis of symmetry, and about a perpendicular axis through the center of mass.

**19.** How many yards of thread 0.03 inch in diameter can be wound on the spool shown in Fig. 5.29?

**20.** Given that the volume of a cone is one-third the area of the base times the height, locate by Pappus' theorem the centroid of a right triangle whose legs are of lengths $a$ and $b$.

**21.** Prove that Pappus' second theorem holds even if the axis of revolution intersects the surface, provided that we take as volume the difference in the volumes generated by the two parts into which the surface is divided by the axis. What is the corresponding generalization of the first theorem?

**22.** Find the center of mass of a wire bent into a semicircle of radius $a$. Find the three radii of



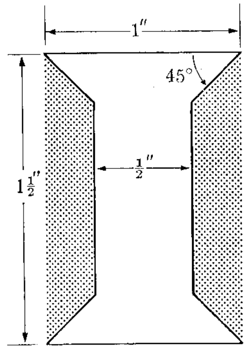


**Fig. 5.29** How much thread can be wound on this spool?

gyration about $x$-, $y$-, and $z$-axes through the center of mass, where $z$ is perpendicular to the plane of the semicircle and $x$ bisects the semicircle. Use your ingenuity to reduce the number of calculations required to a minimum.

**23.** (a) Find a formula for the radius of gyration of a uniform rod of length $l$ about an axis through one end making an angle $\alpha$ with the rod.
b) Using this result, find the moment of inertia of an equilateral triangular pyramid, constructed out of six uniform rods, about an axis through its centroid and one of its vertices.

**24.** Find the radii of gyration of a plane lamina in the shape of an ellipse of semimajor axis $a$, eccentricity $\varepsilon$, about its major and minor axes, and about a third axis through one focus perpendicular to the plane.

**25.** Forces 1 kg-wt, 2 kg-wt, 3 kg-wt, and 4 kg-wt act in sequence clockwise along the four sides of a square $0.5 \times 0.5$ m$^2$. The forces are directed in a clockwise sense around the square. Find the equilibrant.

**26.** Forces 2 lb, 3 lb, and 5 lb act in sequence in a clockwise sense along the three sides of an equilateral triangle. The sides of the triangle have length 4 ft. Find the resultant.

**27.** (a) Reduce the system of forces acting on the cube shown in Fig. 5.30 to an equivalent single force acting at the center of the cube, plus a couple composed of two forces acting at two adjacent corners.
b) Reduce this system to a system of two forces, and state where these forces act.
c) Reduce this system to a single force plus a torque parallel to it.

**28.** A sphere weighing 500 g is held between thumb and forefinger at the opposite ends of a horizontal diameter. A string is attached to a point on the surface of the sphere at the end of a perpendicular horizontal diameter. The string is pulled with a force of 300 g in a direction parallel to the line from forefinger to thumb. Find the forces which must be exerted by forefinger and thumb to hold the sphere stationary. Is the answer unique? Does it correspond to your physical intuition about the problem?